Friends, Foes, and Factions: The math behind signed networks

Fernando Díaz Díaz

Pisan Young Seminars in Applied and NUmerical Mathematics (PYSANUM)

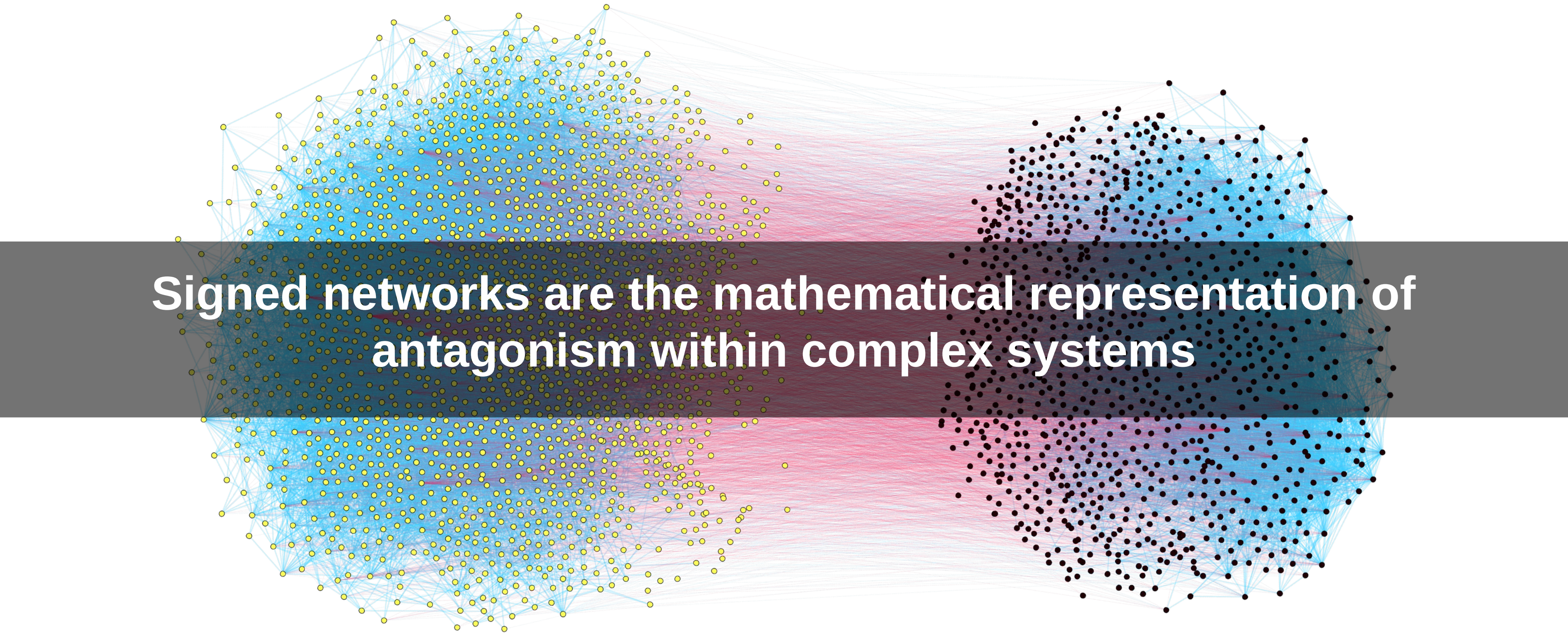
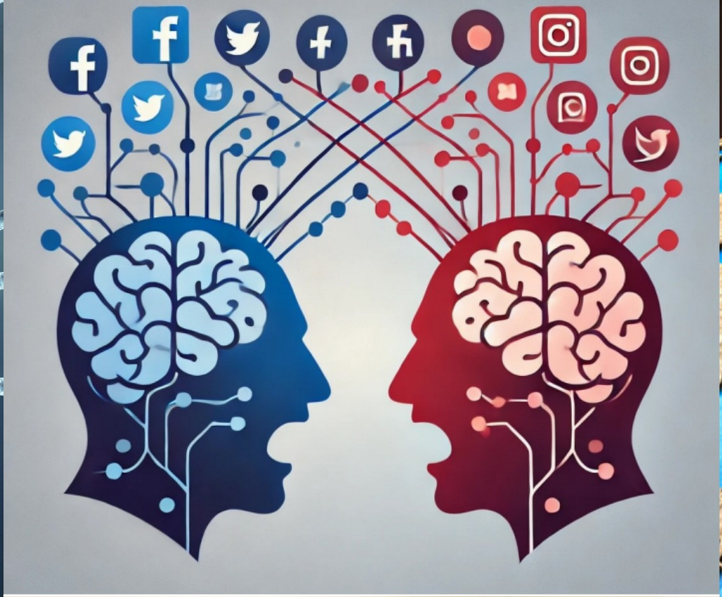


Image credit: Fraxanet et al. (2024) "Unpacking polarization: Antagonism and alignment in signed networks of online interaction." *PNAS Nexus* 3.12 276.




69.928
70.111
20.556
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The role of negative edges

LETTER

nature

Stability criteria for complex ecosystems

Stefano Allesina[1,2] & Si Tang[1]

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BMC Systems Biology

BioMed Central

Research article

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The regulatory network of *E. coli* metabolism as a Boolean dynamical system exhibits both homeostasis and flexibility of response

Areejit Samal[1] and Sanjay Jain*[1,2,3]

The role of negative edges

nature COMMUNICATIONS

ARTICLE

https://doi.org/10.1038/s41467-019-10548-8 OPEN

Structural balance emerges and explains performance in risky decision-making

Omid Askarisichani[1], Jacqueline Ng Lane[2], Francesco Bullo[3,4], Noah E. Friedkin[3,5], Ambuj K. Singh[1] & Brian Uzzi[6,7]

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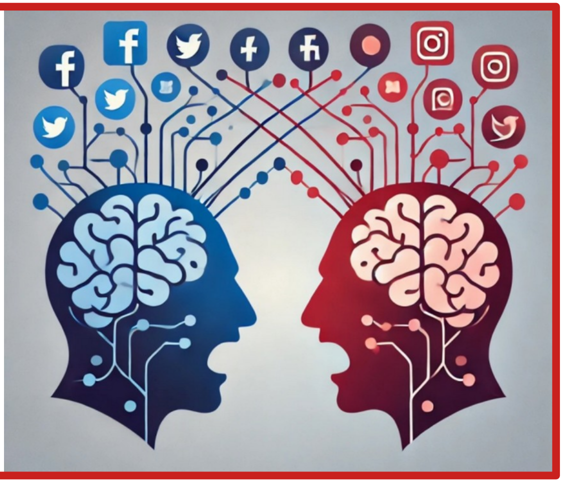
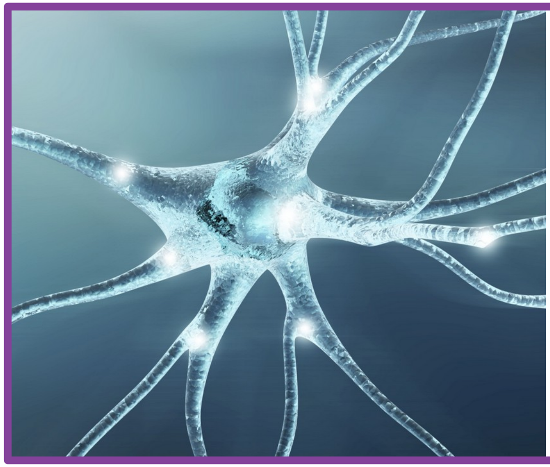
PNAS RESEARCH ARTICLE | SOCIAL SCIENCES

Triadic influence as a proxy for compatibility in social relationships

Miguel Ruiz-García[a,b,c,2,1], Juan Ozaita[c,1], María Pereda[b,d], Antonio Alfonso[e], Pablo Brañas-Garza[e], José A. Cuesta[b,c,f], and Angel Sánchez[b,c,f]



The role of negative edges



PNAS

The human brain is intrinsically organized into dynamic, anticorrelated functional networks

Michael D. Fox*†, Abraham Z. Snyder*‡, Justin L. Vincent*, Maurizio Corbetta‡, David C. Van Essen§, and Marcus E. Raichle*‡§¶

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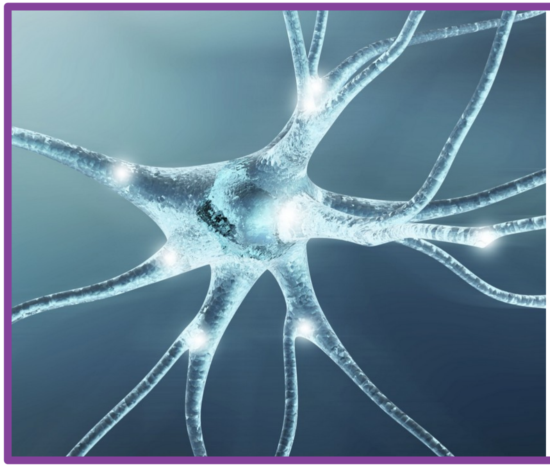

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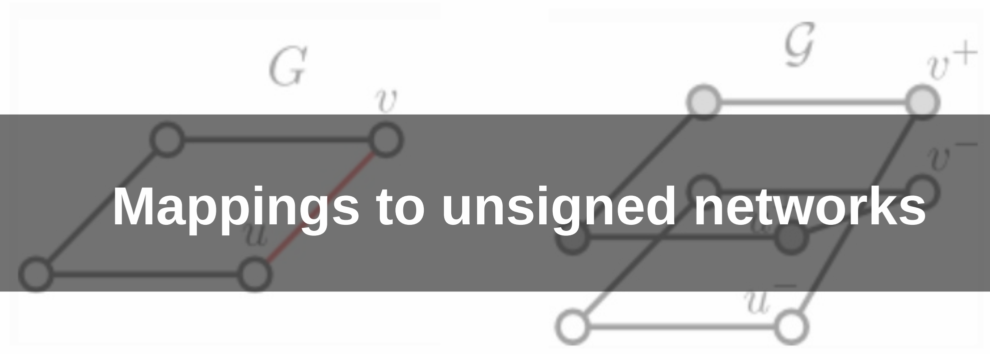
Power and Interdependence

ROBERT O. KEOHANE AND JOSEPH S. NYE, JR

Overview

Structural balance & signed metrics



Mappings to unsigned networks

Dynamics on signed networks

Detection of communities and factions

Images: Gava et al, Science 385.6713 1120-1127; Babul and Lambiotte, Comm. Phys. 7.1, 8;

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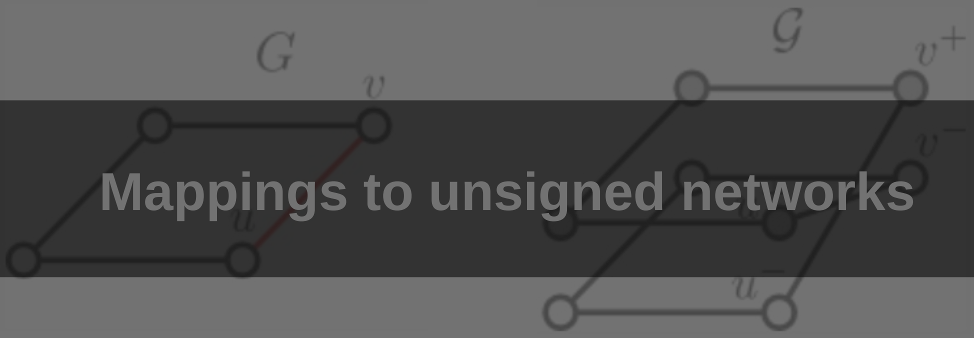
Structural balance & signed metrics



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Mathematical representation of a signed network

Definition 1: a **signed network** is a triple $G = (V,E,\sigma)$, where V is the set of nodes, E is the set of edges, and $\sigma : E \rightarrow \{+1,-1\}$ is a function that assigns a sign (+ or -) to each edge.

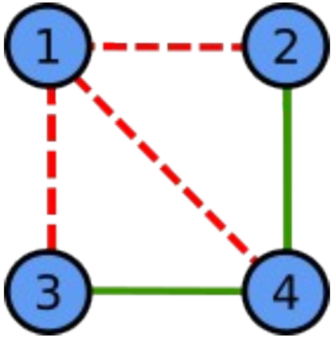
Zaslavsky, T. (1982). “Signed graphs”. *Discrete Applied Mathematics* 4, 47-74.

Signed networks

Mathematical representation of a signed network

Definition 1: a **signed network** is a triple G = (V,E,σ), where V is the set of nodes, E is the set of edges, and $\sigma : E \rightarrow \{+1,-1\}$ is a function that assigns a sign (+ or -) to each edge.

Definition 2: the **adjacency matrix A** of a signed network is a square matrix with elements $A_{ij} = \sigma(e)$ when e=(i,j), and zero otherwise.



$$A = \begin{pmatrix} 0 & -1 & -1 & -1 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{pmatrix}$$

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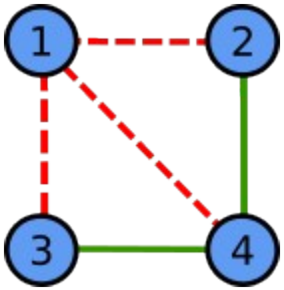
The walk lemma for signed networks

Definition 3: a **walk** is an ordered sequence of (not necessarily different) edges, where consecutive edges are incident to the same node. The **sign** of a walk is the product of the signs of its edges.



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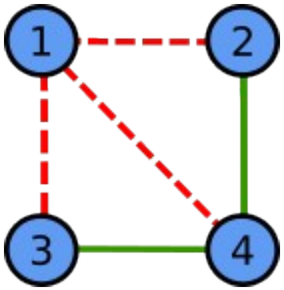
Lemma 4 (walk lemma): the (i,j)th element of the k-th power of the adjacency matrix, $(A^k)_{ij}$, counts the difference between the number of positive walks and the number of negative walks of length k between nodes i and j.



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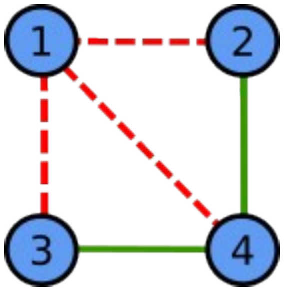

$$A^3 = \begin{pmatrix} 4 & -5 & -5 & -5 \\ -5 & 2 & 2 & 5 \\ -5 & 2 & 2 & 5 \\ -5 & 5 & 5 & 4 \end{pmatrix}$$

$$W_1 = \{e_{12}, e_{24}, e_{43}\}$$
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$$W_4 = \{e_{13}, e_{34}, e_{43}\}$$
$$W_5 = \{e_{13}, e_{31}, e_{13}\}$$

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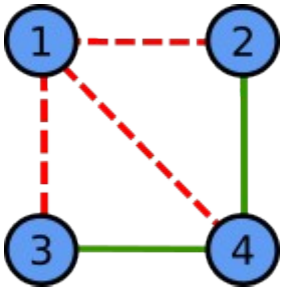

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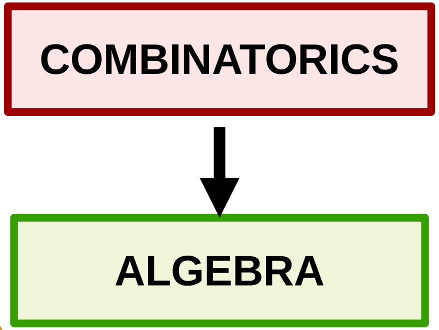

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Structural balance

Harary's theory

Definition 5: A cycle or closed walk is **balanced** or positive if it contains an even number of negative edges.

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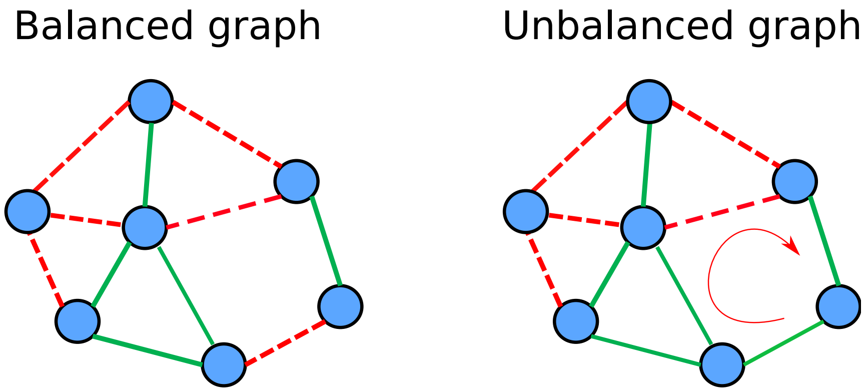
- Cartwright, D., & Harary, F. (1956). “Structural balance: a generalization of Heider's theory”. *Psychological review*, *63*(5), 277.

Structural balance

Harary's theory

Definition 5: A cycle or closed walk is **balanced** or positive if it contains an even number of negative edges.

Definition 6: A signed network is **balanced** if every cycle within it is balanced. Otherwise, the network is **unbalanced**.



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Balance theorems

Theorem 7 (Harary): a graph is balanced iff the node set can be split into two **balanced factions**, such that:

- Links within each balanced faction are positive, and
- Links between different balanced factions are negative.

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Balance theorems

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Theorem 8 (Acharya): a signed graph with adjacency matrix A is balanced if and only if A and |A| have the same spectrum.

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- Acharya (1980). “Spectral criterion for cycle balance in networks”. *Journal of Graph Theory*, 4(1), 1-11.
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Balance indices

How close is a network to a perfectly balanced state?

Balance indices

How close is a network to a perfectly balanced state?

Motif-based approach: count balanced triangles or squares.

- Cartwright and Harary (1956), *Psychological Review* 63.5, 277

Frustration-based approach (spin glass theory): count frustrated edges.

- Aref and Wilson (2019), *Journal of Complex Networks* 7.2, 163–189

Dynamics-based approach: convergence to stationary state of a diffusive process as a proxy for balance.

- Kunegis et al (2010), *Proceedings of the 2010 SIAM international conference on data mining*, pp 559–570

Walk-based approach: count positive and negative walks as a proxy for cycles.

- Estrada and Benzi (2014), *Physical Review E*, 90.4, 042802
- Kirkley, Cantwell, Newman (2019), *Physical Review E* 99.1, 012320

Walk-based balance indices

Local (node-based) levels of balance

Node 3 has medium balance

Walk-based balance indices

Local (node-based) levels of balance

Node 3 has medium balance

Challenges:

- How to enumerate all cycles? → **Walk lemma**
- How to aggregate cycles of different length? → **Weight factor (inverse factorial)**

Walk-based balance indices

Balance index

1) Count balanced and unbalanced closed walks (walk lemma): $(A^k)_{ii}$

Estrada and Benzi. "Walk-based measure of balance in signed networks" PRE 90, 4 (2014): 042802.

Walk-based balance indices

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1) Count balanced and unbalanced closed walks (walk lemma): $(A^k)_{ii}$

2) Include all walk lengths: $\sum_{k=0}^{\infty} \sum_{i=1}^{N} (A^k)_{ii}$

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Balanced graph: κ = 1

Unbalanced graph: 0 < κ < 1

Estrada and Benzi. "Walk-based measure of balance in signed networks" PRE 90, 4 (2014): 042802.

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Balanced graph: $\kappa = 1$
Unbalanced graph: $0 < \kappa < 1$

Also local version!

$$\kappa_i := \frac{(e^A)_{ii}}{(e^{|A|})_{ii}}$$

Diaz-Diaz, Bartesaghi, and Estrada (2024). *Journal of Applied Mathematics and Computing*, 1–24.

Estrada and Benzi. "Walk-based measure of balance in signed networks" PRE 90, 4 (2014): 042802.

Communicability

Up to now, we have counted **closed walks**, to quantitatively measure **balance.**

Now, we will enumerate **open walks**, to quantitatively measure **effective alliances and enmities.**

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Communicability matrix:

$$\Gamma_{ij} = \sum_{k=0}^{\infty} \frac{(A^k)_{ij}}{k!} = (e^A)_{ij}$$

- Diaz-Diaz and Estrada. "Signed graphs in data sciences via communicability geometry." Information Sciences 710 (2025): 122096.

The distance problem in signed graphs

Definition 11: a distance is a function d that satisfies the following axioms:

1) Non-negativity: $d(i,j) \geq 0$

2) Identity of indiscernibles: $d(i,j)=0$ iff $i=j$

3) Symmetry: $d(i,j)=d(j,i)$

4) Triangle inequality: $d(i,j)+d(j,k) \geq d(i,k)$

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How can we define a well-defined distance on a signed graph?

Example 12: minimum-weight distance in a signed graph:

~~1) Non-negativity~~
~~2) Identity of indiscernibles~~
3) Symmetry
~~4) Triangle inequality~~

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How can we define a well-defined distance on a signed graph?

Example 13: communicability distance in a signed graph:

$$\xi_{ij} = \sqrt{\Gamma_{ii} + \Gamma_{jj} - 2\Gamma_{ij}}$$

Very central nodes increase the distance

Effective enemies (negative comm) increase the distance

The distance problem in signed graphs

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ξ is a Euclidean distance, even when the graph is signed

Example 13: communicability distance in a signed graph:

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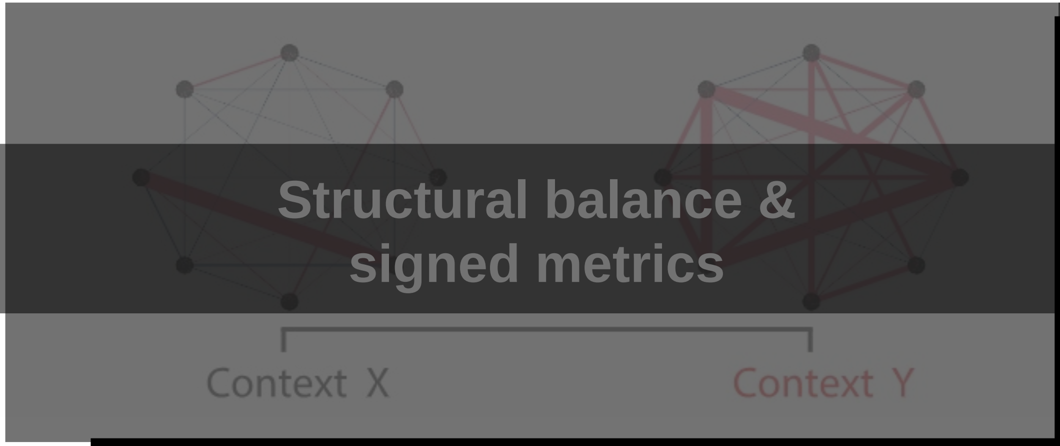
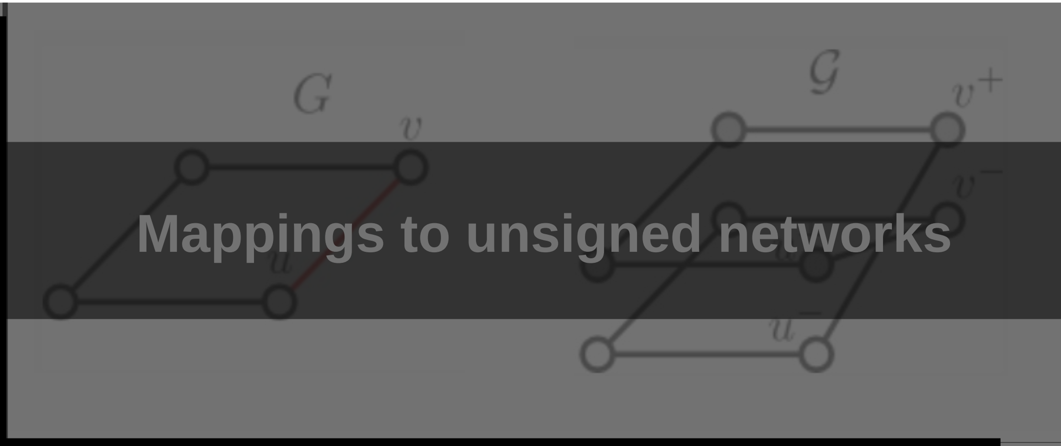
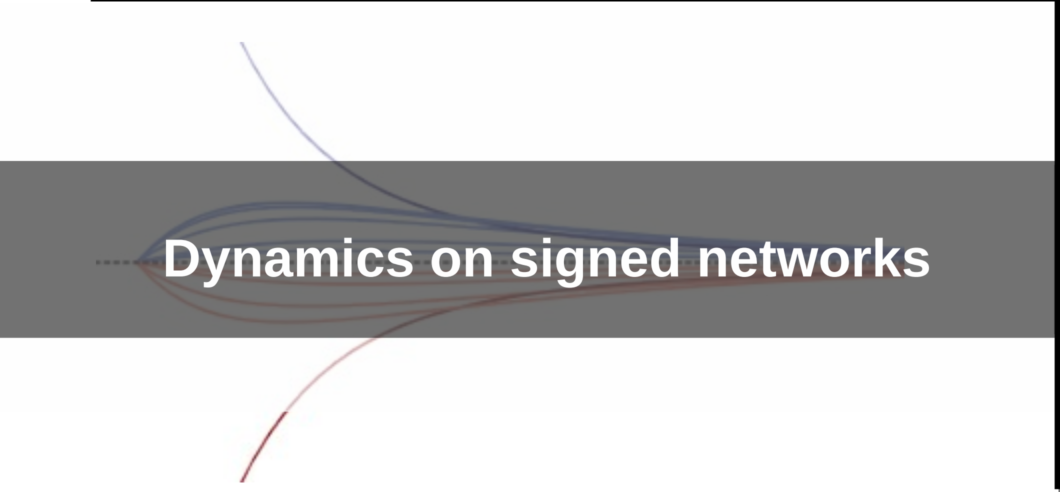
Overview



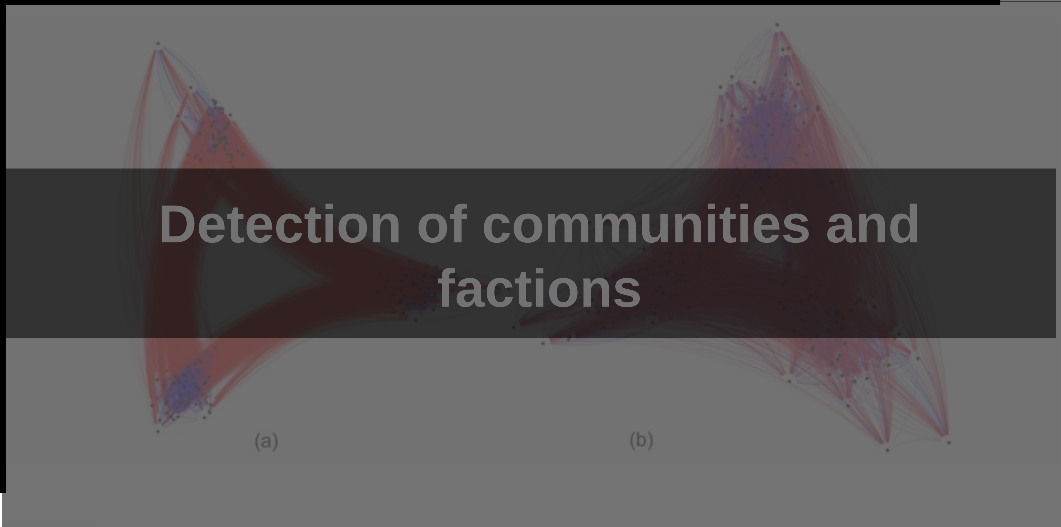
Structural balance & signed metrics



Mappings to unsigned networks



Dynamics on signed networks



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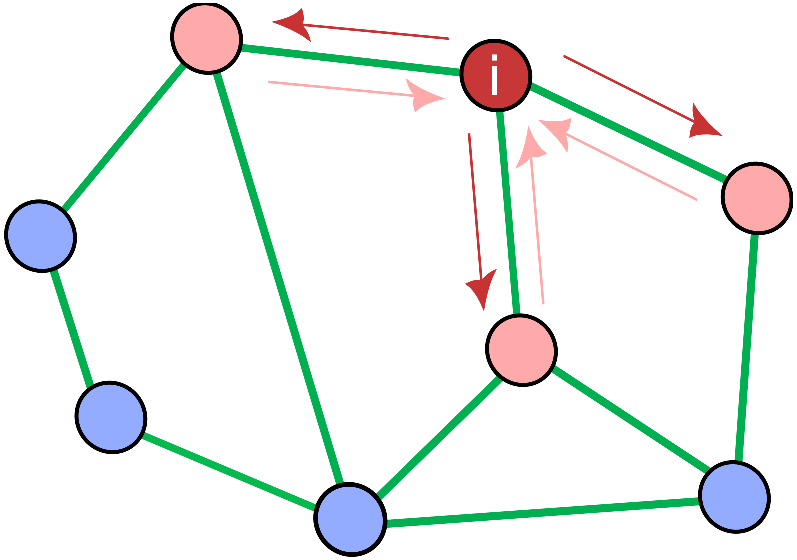
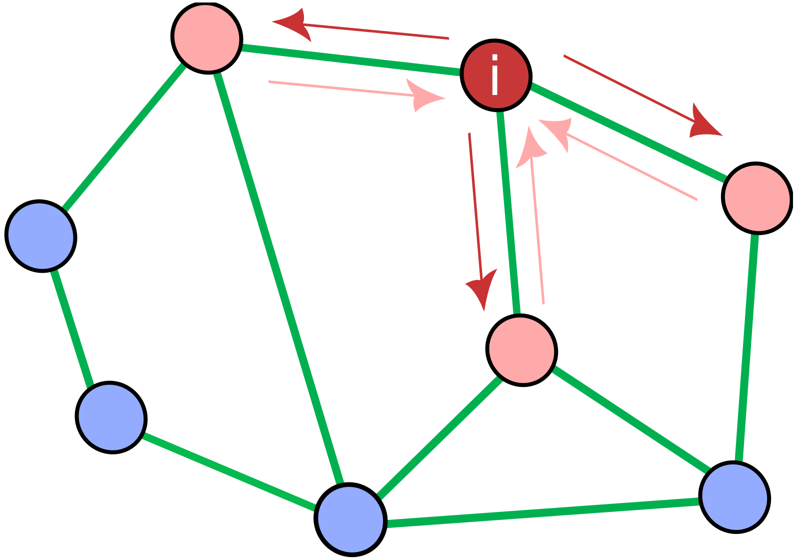
Diffusion in networks



Masuda, N., Porter, M. A., & Lambiotte, R. (2017). Random walks and diffusion on networks. Physics reports, 716, 1-58.

Diffusion in networks

$$x_i(t+1) = x_i(t) - \sum_{j \leftarrow i} x_i(t) + \sum_{j \rightarrow i} x_j(t)$$



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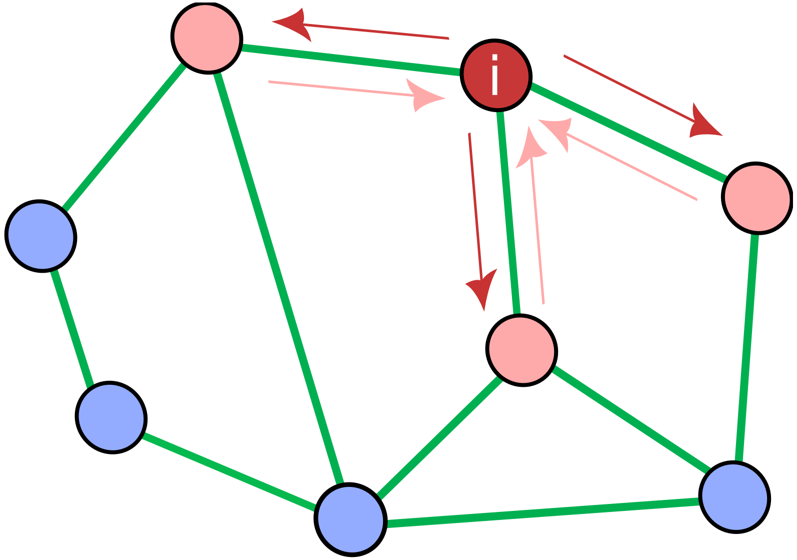

$$x_i(t+1) = x_i(t) - \sum_{j \leftarrow i} x_i(t) + \sum_{j \rightarrow i} x_j(t)$$

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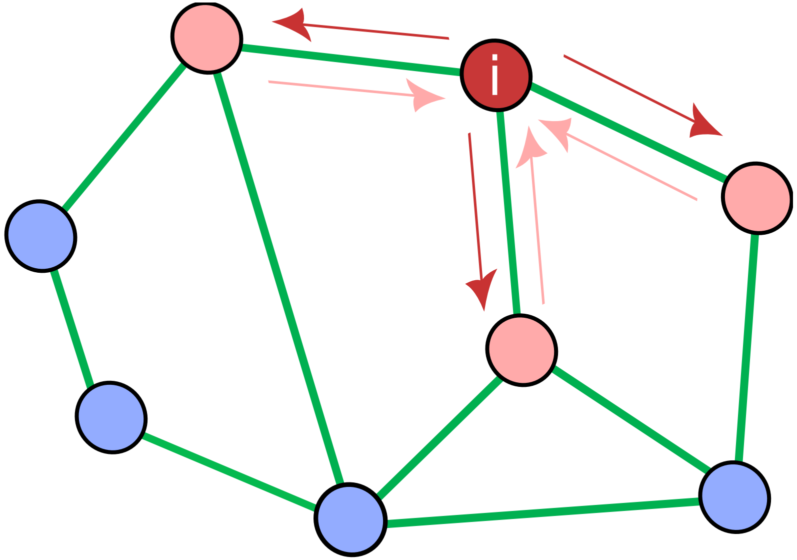

$$x_i(t+1) = x_i(t) - \sum_{j \leftarrow i} x_i(t) + \sum_{j \rightarrow i} x_j(t)$$

$$\dot{x}(t) = -(K - A)x(t)$$

$$\dot{x}(t) = -Lx(t)$$

Diffusion equation for graphs

Masuda, N., Porter, M. A., & Lambiotte, R. (2017). Random walks and diffusion on networks. Physics reports, 716, 1-58.

Signed Laplacian

Definition 14: the **Laplacian operator** L of an unsigned network is given by:

$$L_{ij} = \begin{cases} k_i & \text{if} \quad i = j \\ -A_{ij} & \text{otherwise} \end{cases}$$

where $k_i = \sum_j A_{ij}$

Kunegis, Schmidt, Lommatzsch, Lerner, De Luca, & Albayrak. (2010). Spectral analysis of signed graphs for clustering, prediction and visualization.
In Proceedings of the 2010 SIAM international conference on data mining (pp. 559-570). SIAM

Signed Laplacian

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Theorem 16: the Laplacian operator of a signed graph is a **positive semidefinite**. Moreover, L has a null eigenvalue if and only if the graph is balanced.

Kunegis, Schmidt, Lommatzsch, Lerner, De Luca, & Albayrak. (2010). Spectral analysis of signed graphs for clustering, prediction and visualization. In Proceedings of the 2010 SIAM international conference on data mining (pp. 559-570). SIAM

Diffusion on signed networks

Linear dynamics (**diffusion**):

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In the stationary state:

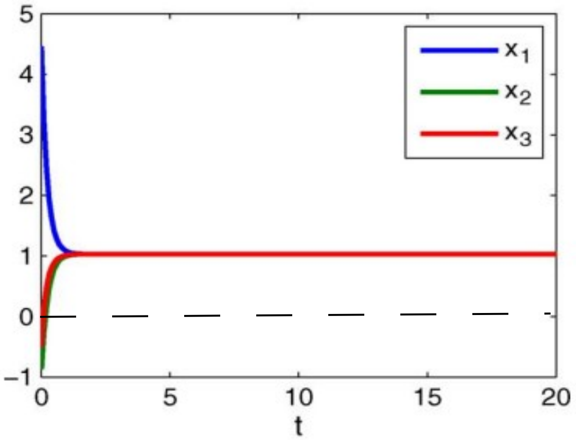
$$\lim_{t\to\infty} x_i(t) = \begin{cases} 0 & \text{if} \quad \lambda_0 > 0 \\ C\psi_0 & \text{otherwise} \end{cases} \qquad (\psi_0)_i = \begin{cases} +1 & \text{if} \quad i \in G_1 \\ -1 & \text{if} \quad i \in G_2 \end{cases}$$

Diffusion on signed networks

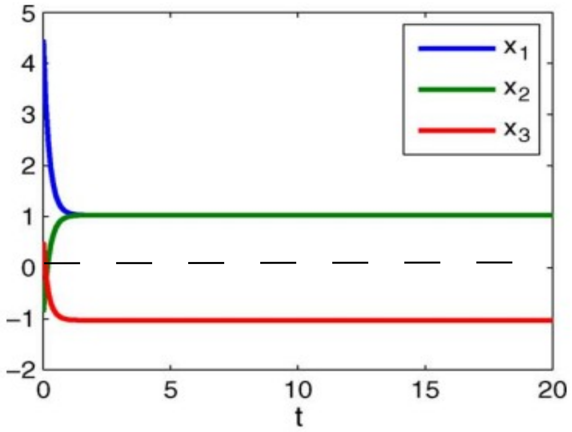
Theorem 17: the stationary state of a diffusive process on a signed network depends on the structural balance of the network. In particular, the stationary state is:

- **Consensus** if the network is unsigned.
- **Agreed dissensus** if the network is balanced.
- **Absence of opinions** if the network is unbalanced.

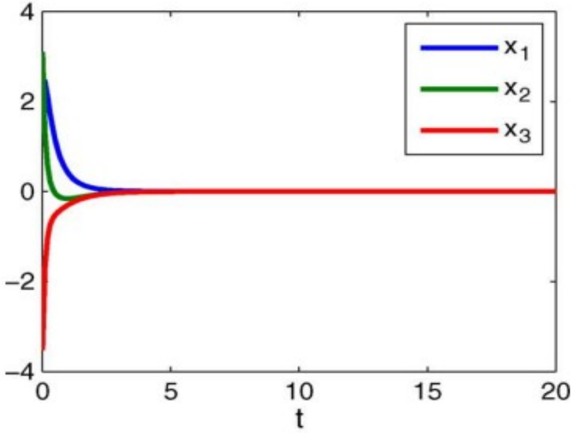
Consensus



Agereed dissensus



Absence of opinions



Altafini, C. (2012). Consensus problems on networks with antagonistic interactions. IEEE transactions on automatic control, 58(4), 935-946.

Diffusion on signed networks

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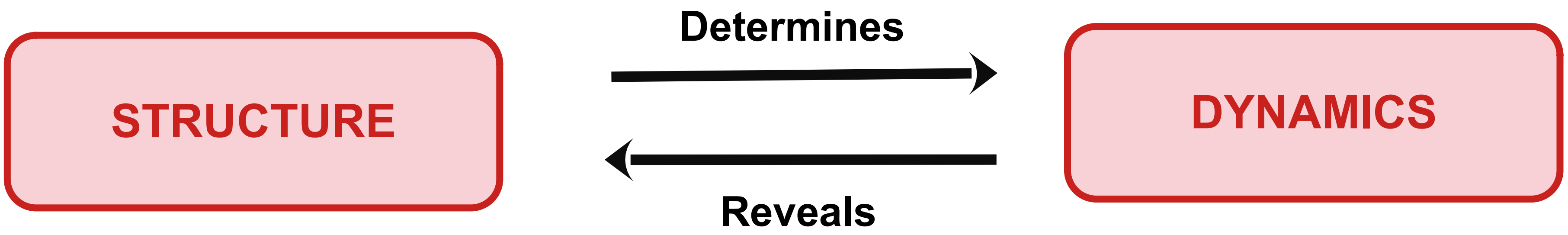

Altafini, C. (2012). Consensus problems on networks with antagonistic interactions. IEEE transactions on automatic control, 58(4), 935-946.

Some essential references

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- Thomas Zaslavsky. “Signed Graphs”. Discrete Applied Mathematics 4 (1982), pages 47–74.
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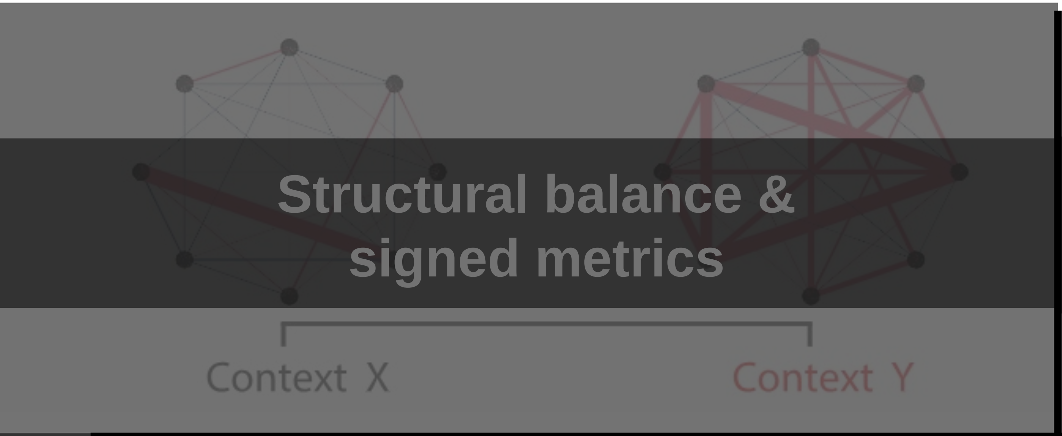
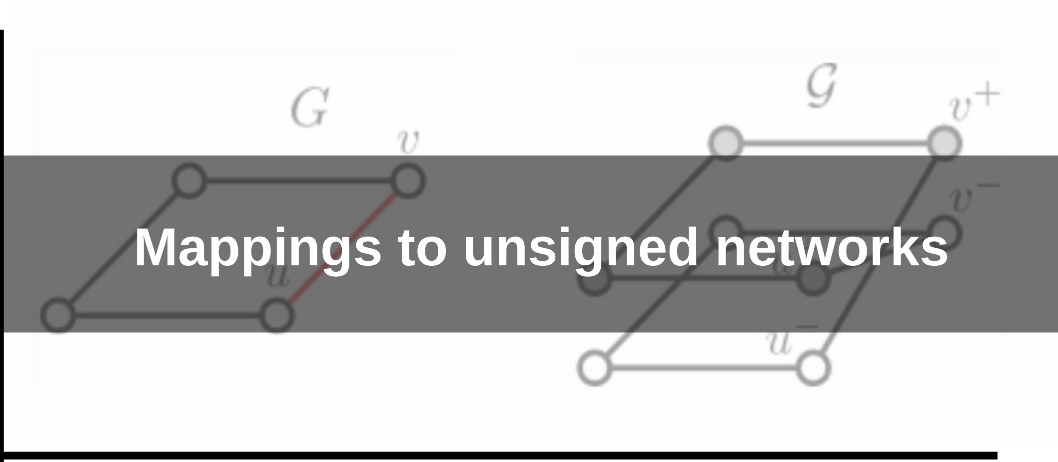
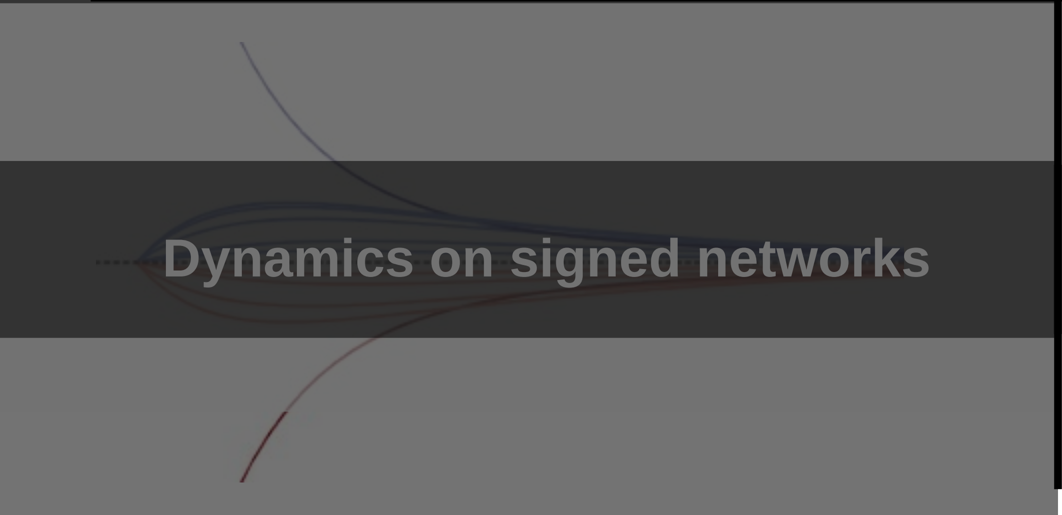
- Claudio Altafini. “Consensus problems on networks with antagonistic interactions”. IIEEE transactions on automatic control 58.4 (2012), pages 935–946.

Overview

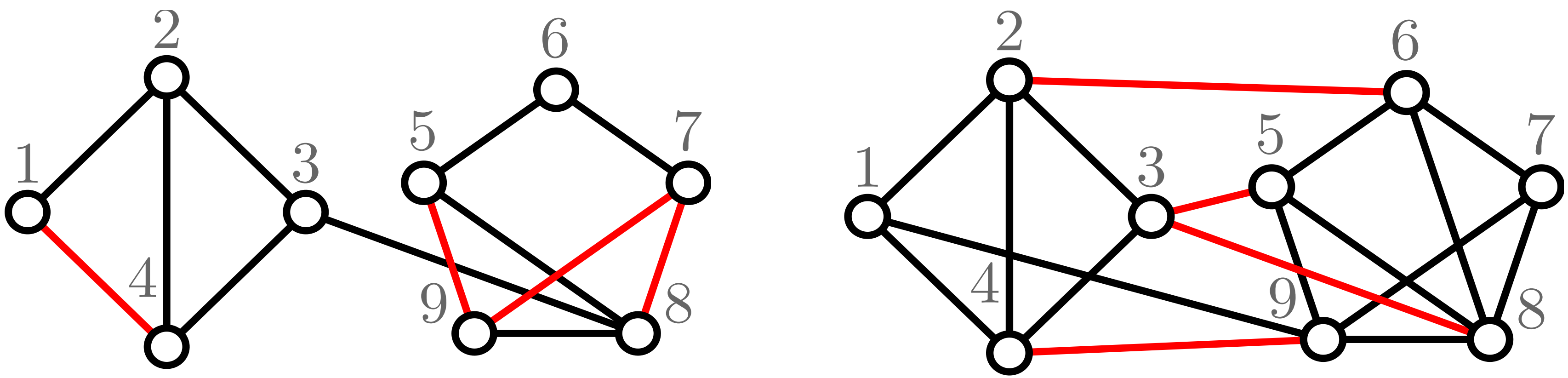






Images: Gava et al, Science 385.6713 1120-1127; Babul and Lambiotte, Comm. Phys. 7.1, 8;

Q: which signed network has mesoscale structure?



Communities and factions

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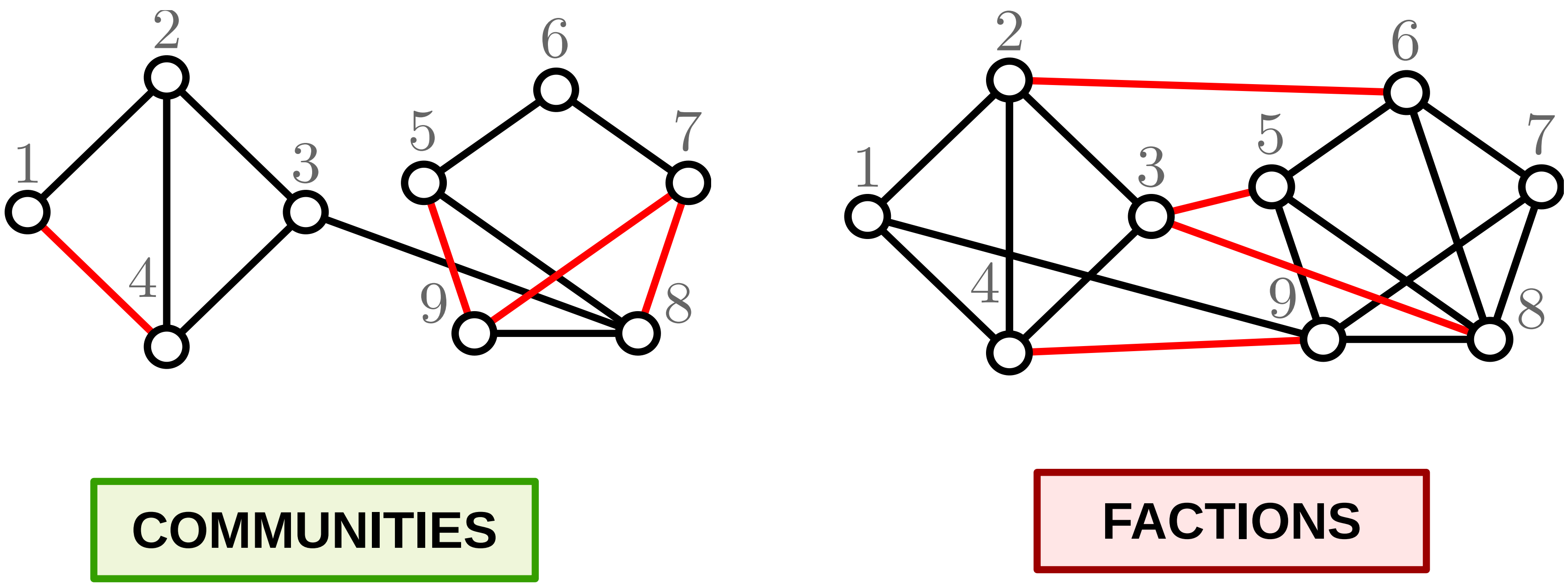

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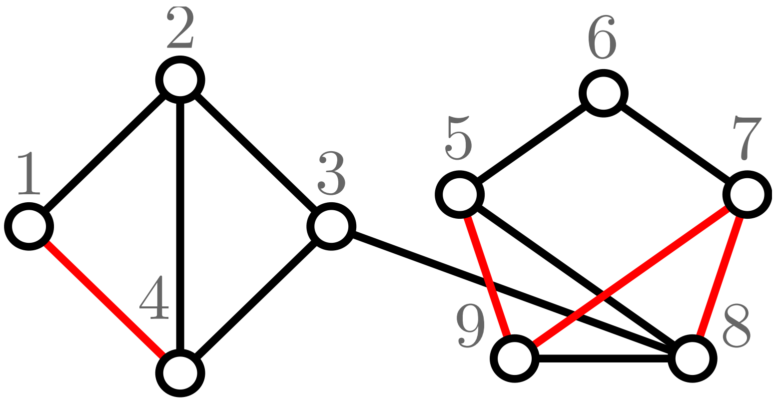
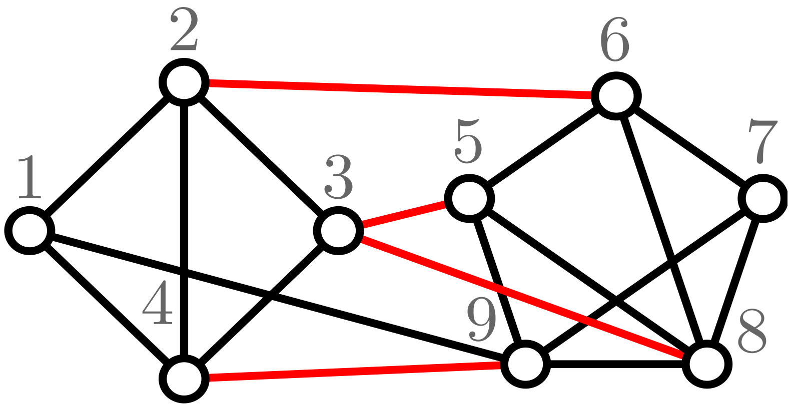
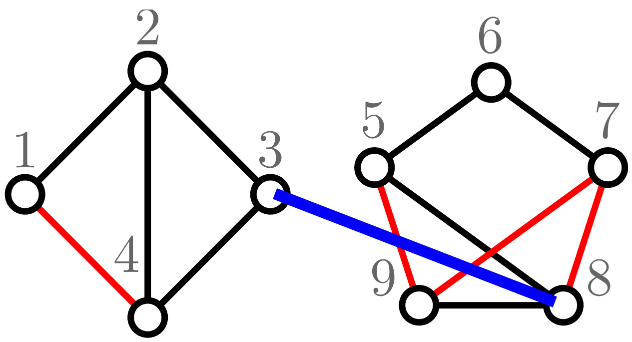




Communities and factions

Definition (Cut set)

For a partition $V = U_1 \cup U_2$, the cut-set $C(U_1, U_2)$ is the set of edges with one endpoint in U_1 and the other in U_2.
The edge connectivity $\kappa_e(G)$ is the size of a smallest cut-set of G.



Communities have small edge connectivity

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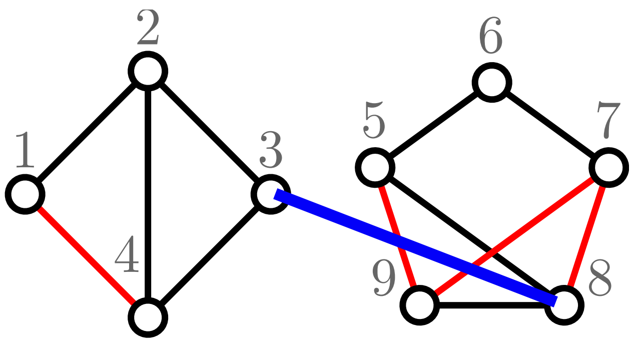
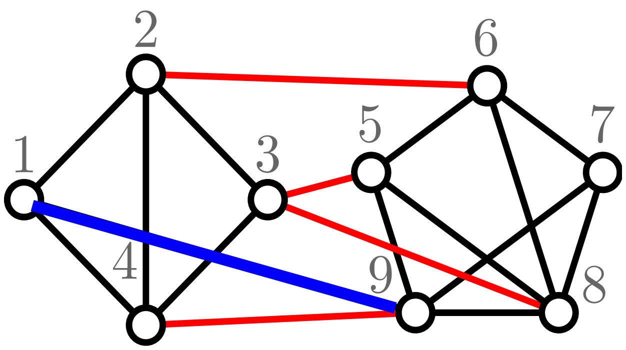
Definition (Frustration set)

A frustration set is a set of edges whose removal makes the signed graph balanced.
The frustration index $\phi(G)$ is the size of a smallest frustration set of G.



Communities have small edge connectivity



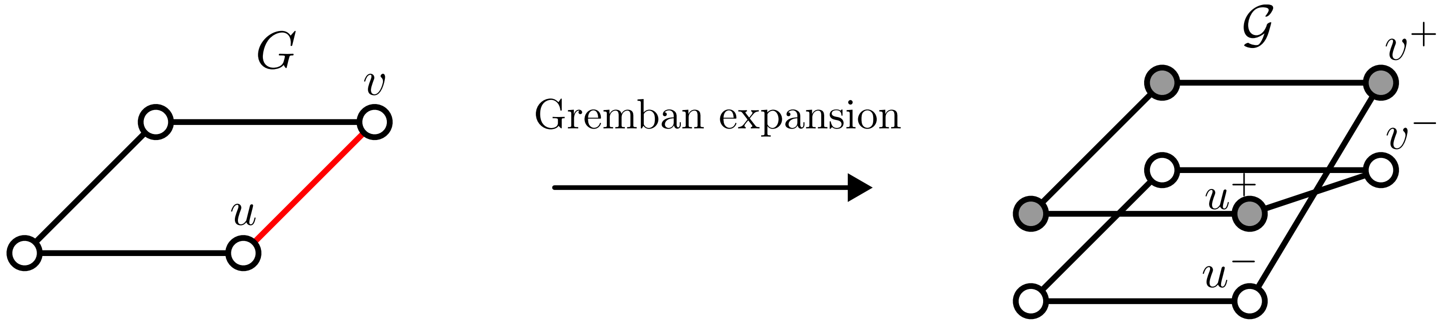
Factions have small frustration index

Gremban expansion

Definition (Gremban expansion)

The *Gremban expansion* of a signed graph G is the unsigned graph $\mathcal{G}$ with $2n$ vertices and $2m$ edges, defined as follows:

- Each node v gets mapped to two polarities v^+ and v^-.
- Positive links (u, v) get mapped to (u^+, v^+) and (u^-, v^-).
- Negative links (u, v) get mapped to (u^+, v^-) and (u^-, v^+).

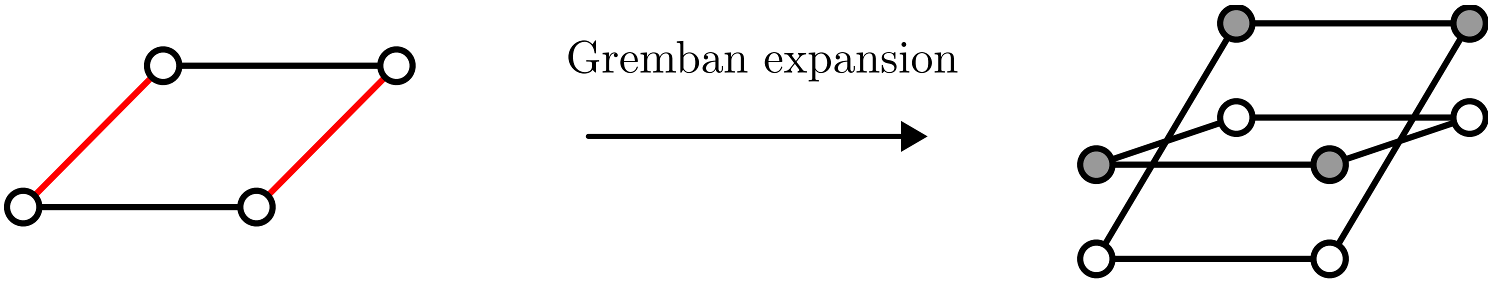


Gremban. Combinatorial preconditioners for sparse, symmetric, diagonally dominant linear systems. PhD thesis, Carnegie Mellon University, 1996.

Gremban expansion

Theorem (1)

A connected signed graph is balanced iff its Gremban expansion is disconnected.



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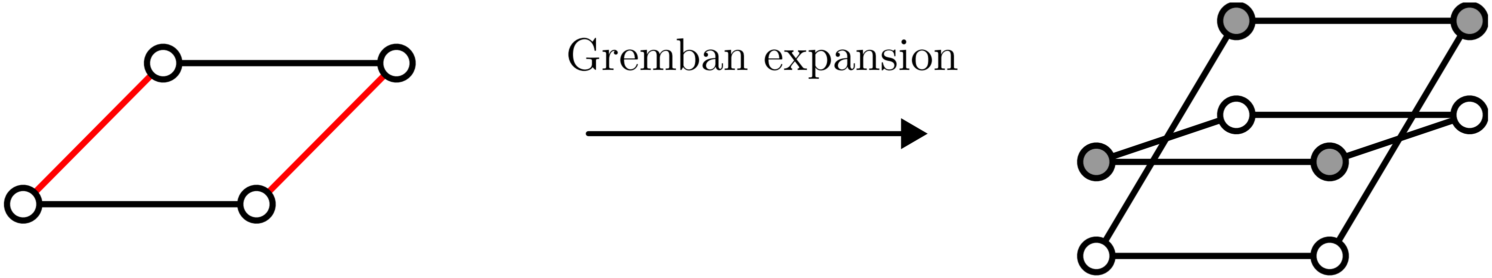
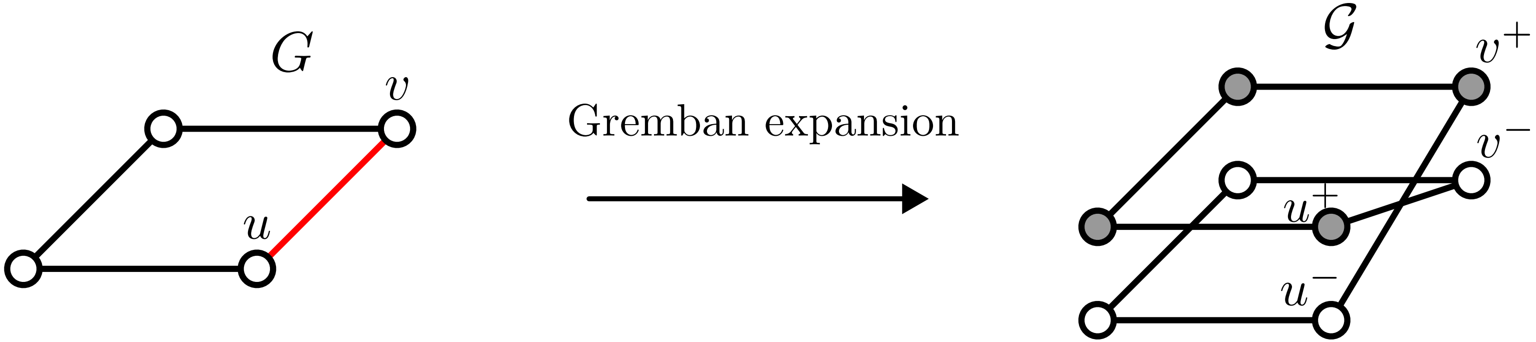


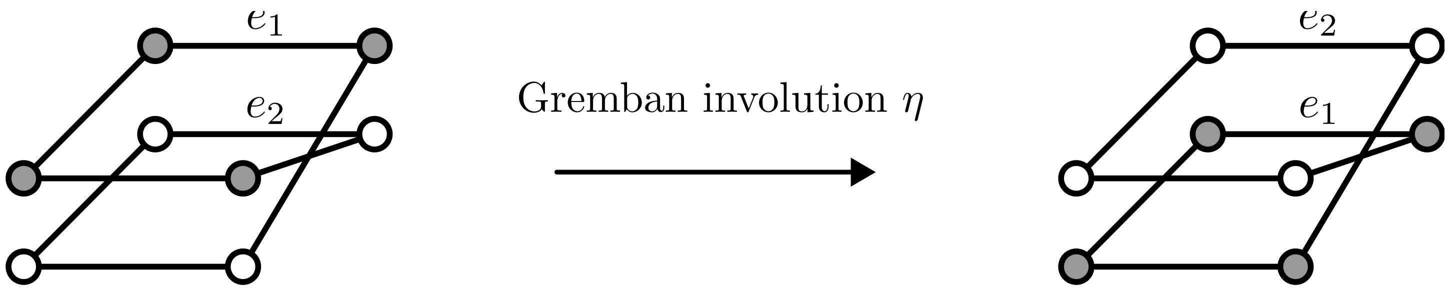


Gremban involution

Definition (Gremban involution)

The *Gremban involution* η swaps the two polarities of every node:

$$\eta(v^+) = v^-, \quad \eta(v^-) = v^+.$$



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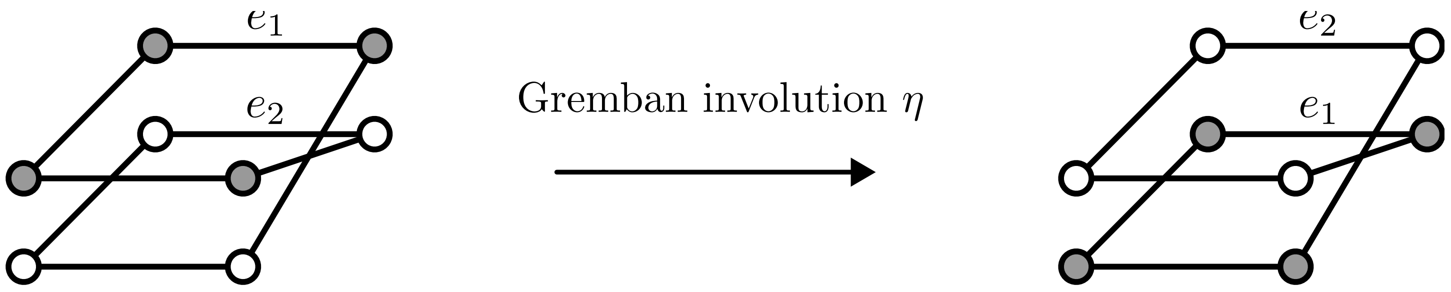
Definition (Gremban symmetry)

- A set (of nodes, edges, or a subgraph) is *Gremban-symmetric* if it is invariant under the involution η, i.e.

$$\eta(\mathcal{X}) = \mathcal{X}.$$

- A bipartition $(\mathcal{U}_1, \mathcal{U}_2)$ of $V(\mathcal{G})$ is *Gremban-symmetric* if $\eta(\mathcal{U}_1) = \mathcal{U}_1$ or $\eta(\mathcal{U}_1) = \mathcal{U}_2$.

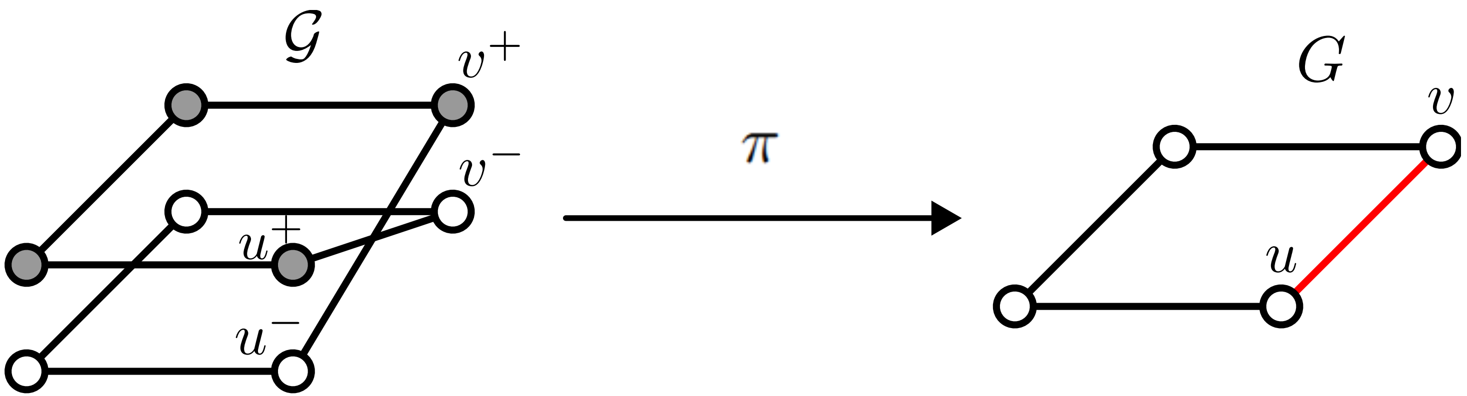


Projecting back to signed graphs

Definition (Projection map)

The *projection* π sends each polarity back to its node:

$$\pi(v^+) = \pi(v^-) = v.$$



Recap

So far, we have:

① A way to operationalize communities and factions (small **cut-sets** and small **frustration sets**).

② An operation, the **Gremban expansion**, that maps a signed graph to an unsigned one.

③ A notion of **symmetry** in the expanded space (invariance under the **involution** η).

④ A way of **projecting** back to the original space, but only for Gremban-symmetric objects.

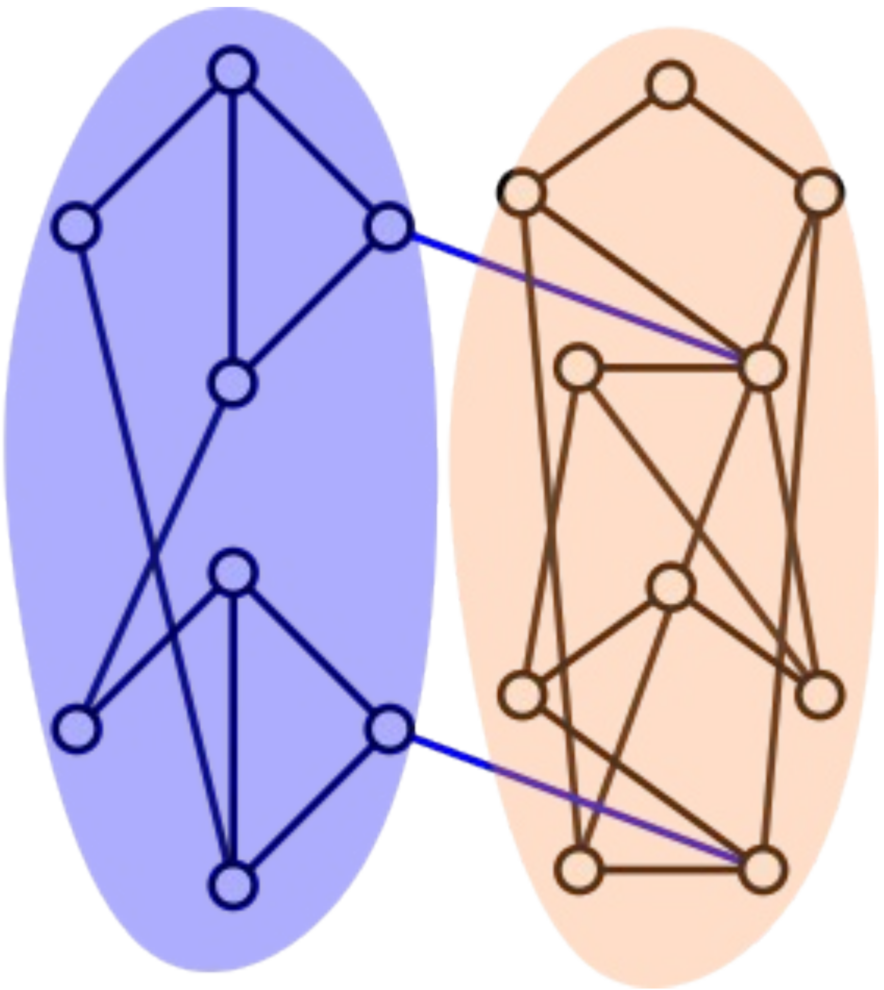
What types of structures structure in the expanded space correspond to communities and factions?

Theorem (3)

Every cut or frustration set in G lifts to a symmetric cut in $\mathcal{G}$. Conversely, every Gremban-symmetric bipartition $(\mathcal{U}_1, \mathcal{U}_2)$ of $\mathcal{G}$ projects to:

$$\begin{cases} \text{cut-sets in } G & \text{if } \eta(\mathcal{U}_1) = \mathcal{U}_1, \\ \text{frustration sets in } G & \text{if } \eta(\mathcal{U}_1) = \mathcal{U}_2. \end{cases}$$

Communities, factions, and Gremban expansions



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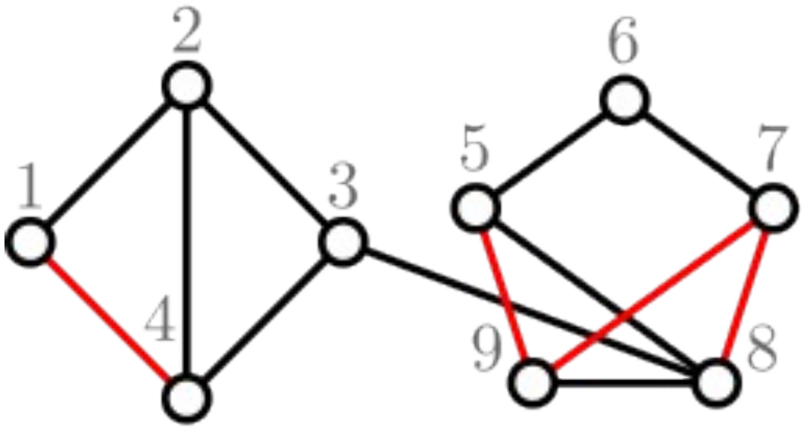

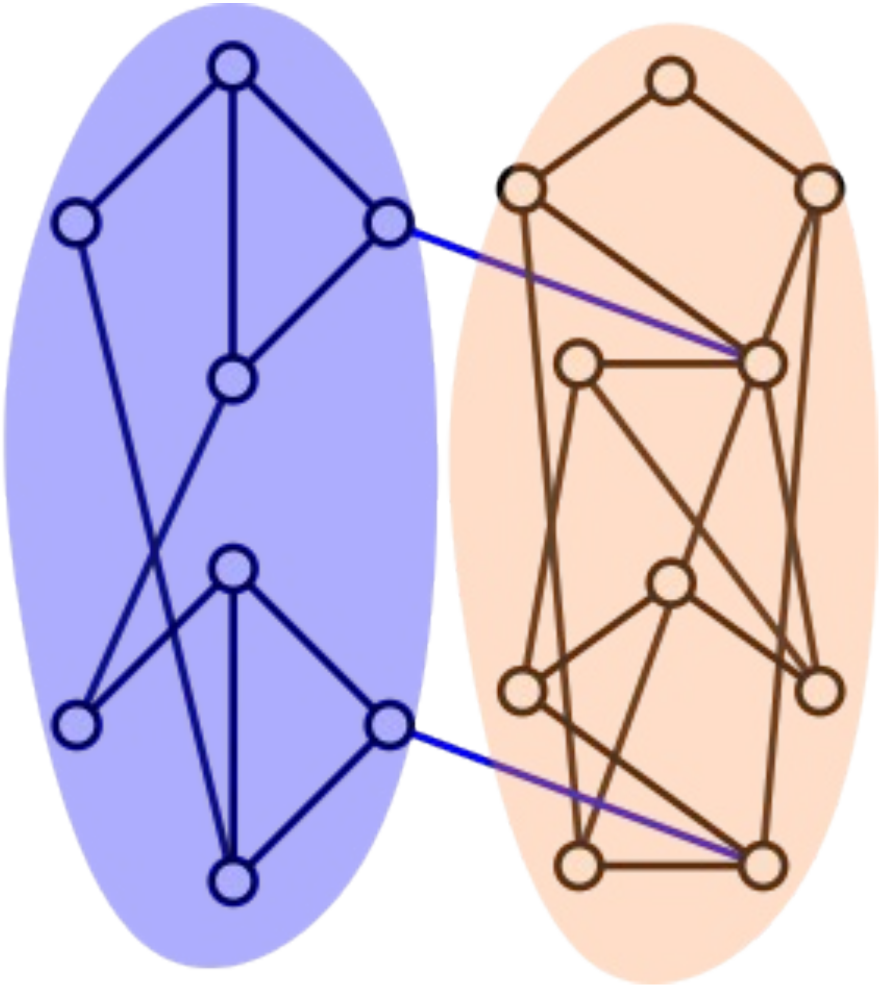

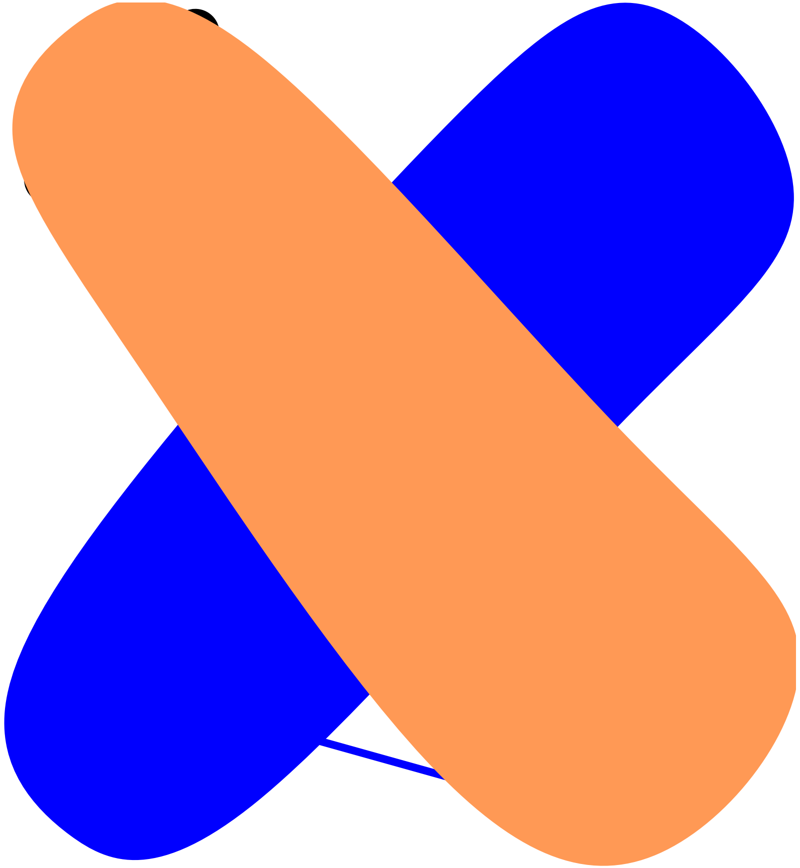

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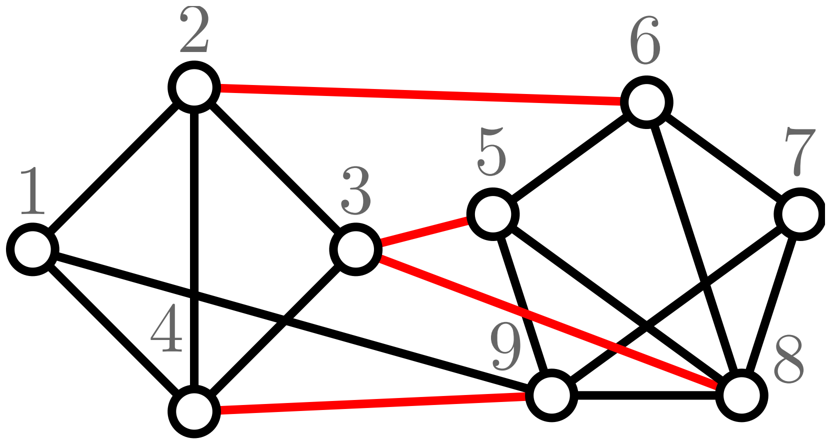
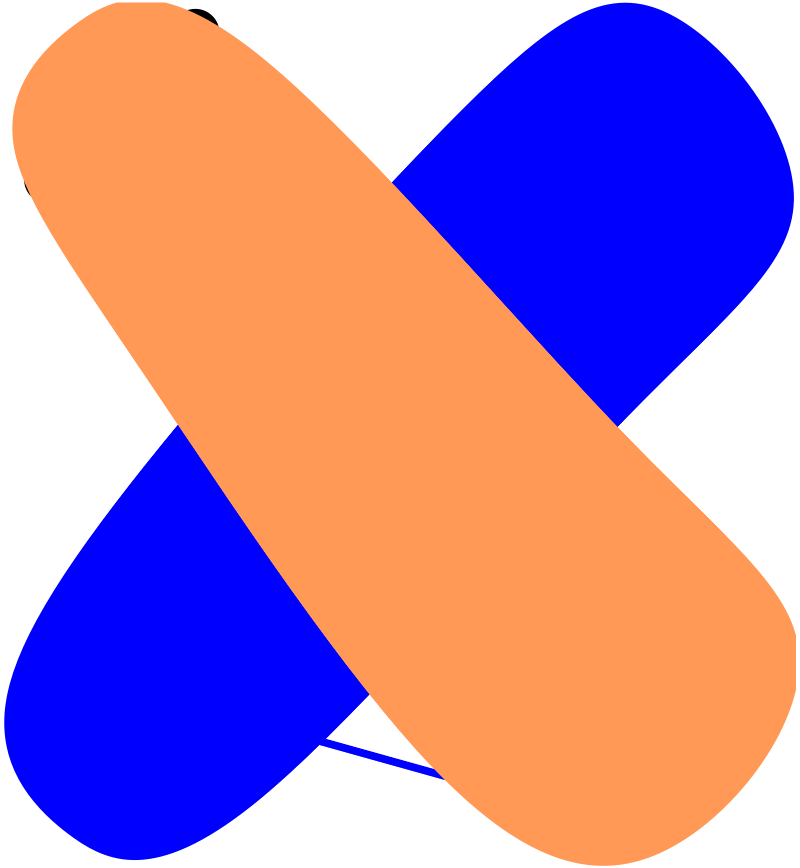
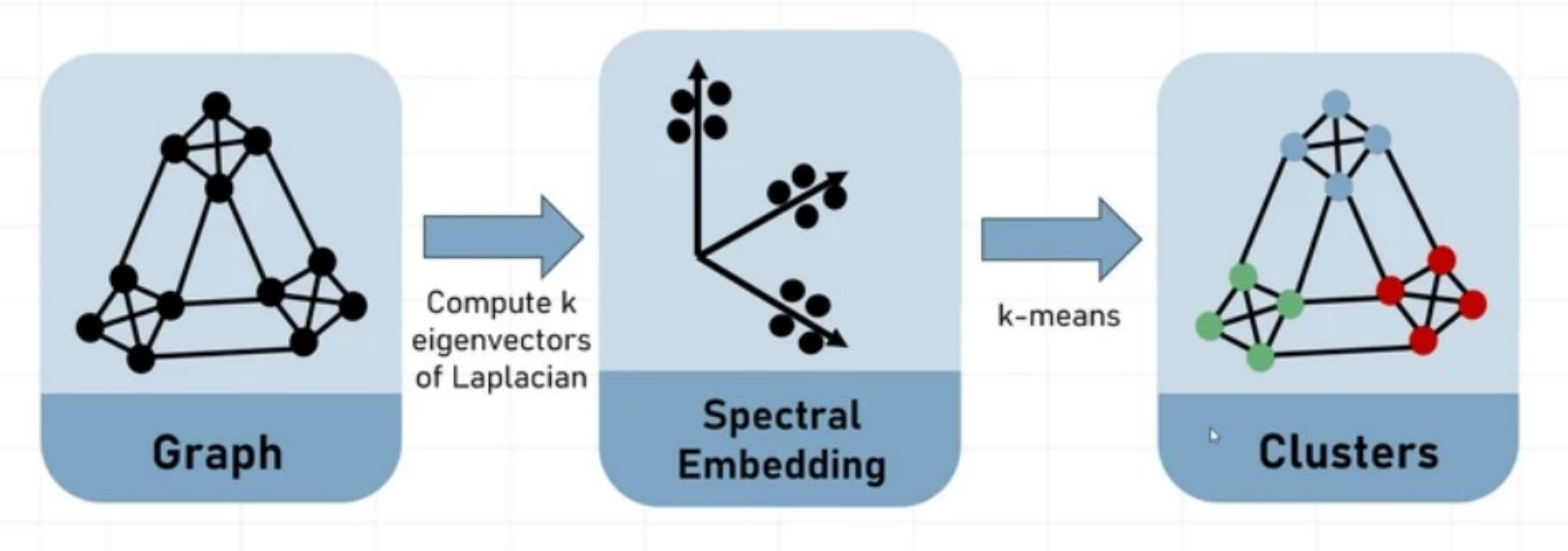




Recap: spectral clustering

Spectral Clustering Principle:
Low eigenvectors of the Laplacian have similar values on well-connected nodes → **eigenvectors of L reveal community structure**



- Ulrike von Luxburg. A tutorial on spectral clustering. Statistics and Computing, 17(4):395–416, 2007.
- Image: Macgregor, Peter. "Fast and simple spectral clustering in theory and practice." Advances in Neural Information Processing Systems 36 (2023): 34410-34425.

Algebraic representation of Gremban expansions

① Signed adjacency matrix. $$A = A^{+} - A^{-}$$

Jerome Kunegis et al. Spectral analysis of signed graphs for clustering, prediction and visualization. In Proceedings of the 2010 SIAM international conference on data mining, pages 559–570. SIAM, 2010.

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② Gremban-expanded adjacency matrix (**non-negative!**).

$$\mathcal{A} = \begin{pmatrix} A^+ & A^- \\ A^- & A^+ \end{pmatrix}$$

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$$\mathcal{A} = \begin{pmatrix} A^+ & A^- \\ A^- & A^+ \end{pmatrix}$$

↓ $L = K - A$

③ Laplacian of the expanded adjacency matrix.

$$\mathcal{L} = \begin{pmatrix} K - A^+ & -A^- \\ -A^- & K - A^+ \end{pmatrix}$$

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Algebraic representation of Gremban expansions

Theorem (4)

Let $L = K - A$ be the signed Laplacian and $\bar{L} = K - |A|$ the unsigned Laplacian. Then, $\mathcal{L} \sim \bar{L} \oplus L$.

$$\mathcal{L} = \mathcal{U} \begin{pmatrix} \bar{L} & 0 \\ 0 & L \end{pmatrix} \mathcal{U}^\top$$

- $(\lambda, x) \in L \Rightarrow (\lambda, (x, -x)) \in \mathcal{L}$,
- $(\mu, y) \in \bar{L} \Rightarrow (\mu, (y, y)) \in \mathcal{L}$.

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Communities → Unsigned topology → Unsigned Laplacian → Symmetric eigenvectors → Gremban-symmetric node partitions

Factions → Signed topology → Signed Laplacian → Antisymmetric eigenvectors → Gremban-antisymmetric node partitions

Spectral clustering in the Gremban expansion

Algorithm:

1. Expand signed graph $G \mapsto \mathcal{G}$.
2. Compute Laplacian $\mathcal{L}$ of $\mathcal{G}$.
3. Extract first non-constant $k-1$ eigenvectors $\{\psi_2, \ldots, \psi_k\}$.
4. Embed nodes in $\mathbb{R}^{k-1}$ and run k-means.
5. Interpret clusters:
   - Symmetric $\rightarrow$ communities
   - Antisymmetric $\rightarrow$ factions inside communities

Diaz-Diaz, Fernando, Karel Devriendt, and Renaud Lambiotte. "Gremban Expansion for Signed Networks: Algebraic and Combinatorial Foundations for Community-Faction Detection." arXiv preprint arXiv:2509.14193 (2025).

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This detects **both communities and factions** and disentangles them in a principled way!

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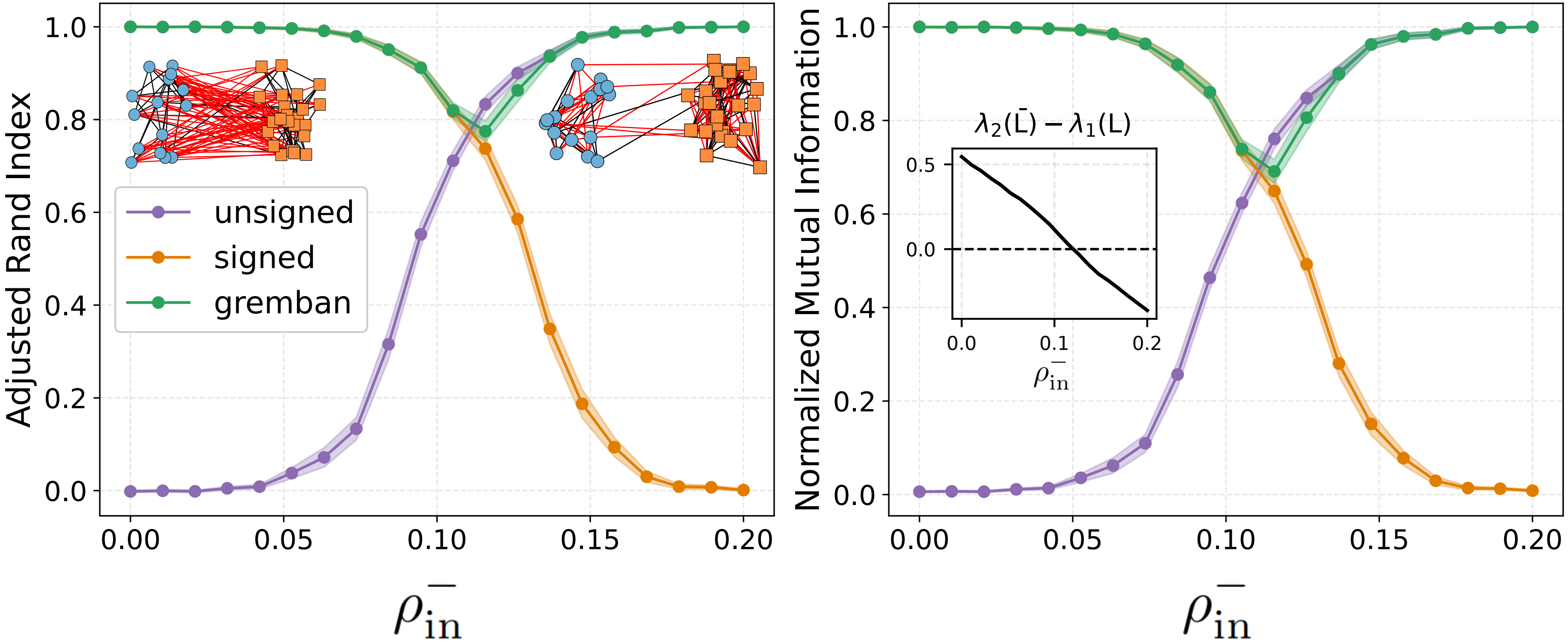

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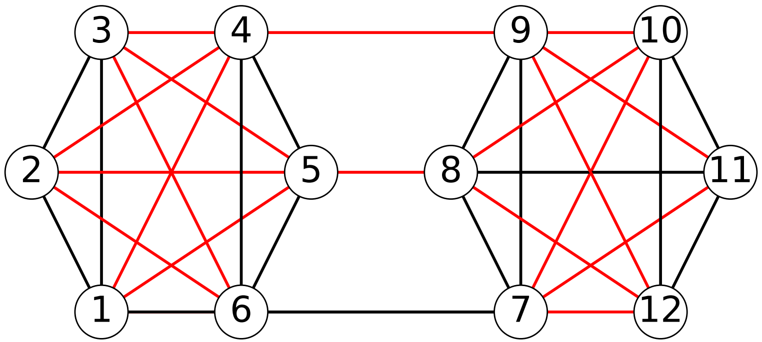
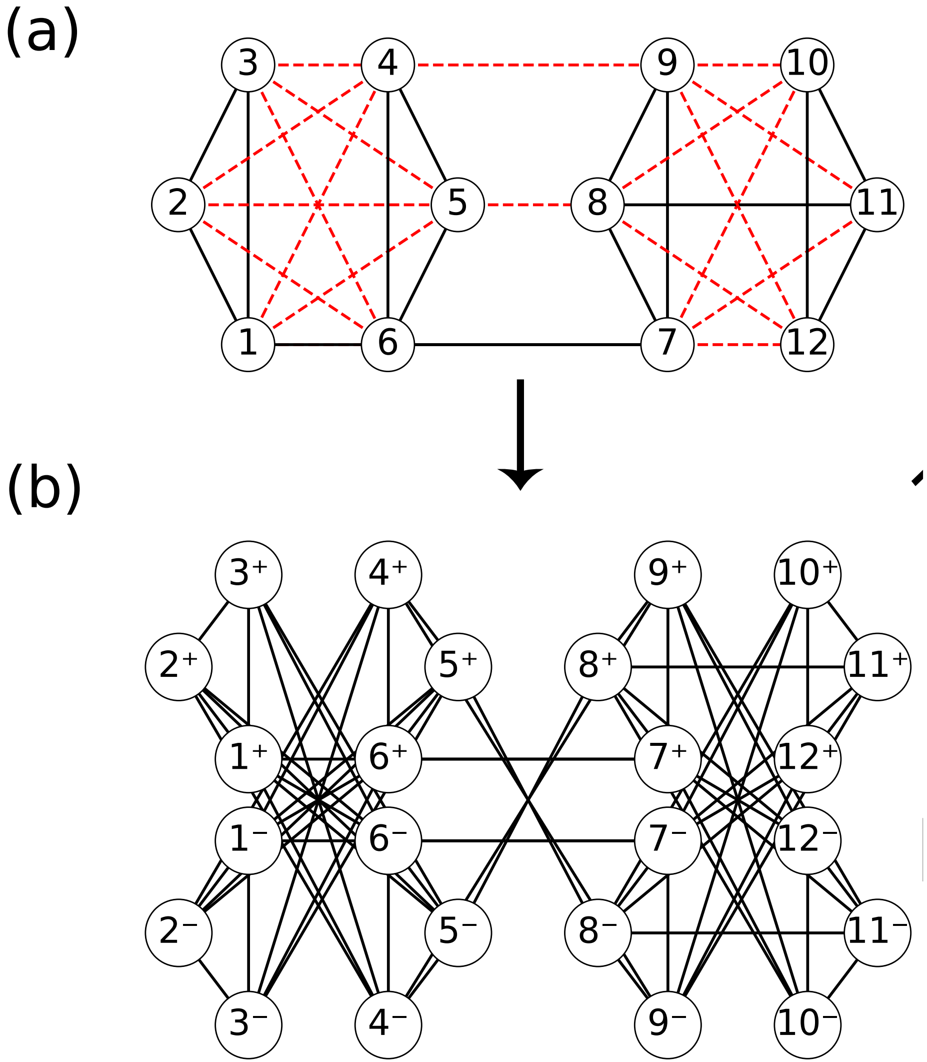
- Fox, Manteuffel, and Sanders. Numerical methods for Gremban's expansion of signed graphs. SIAM Journal on Scientific Computing, 39(5):S945–S968, 2017

(a)




(a)
3
4
9
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2
5
8
11
1
6
7
12
(b)
3⁺
4⁺
9⁺
10⁺
2⁺
5⁺
8⁺
11⁺
1⁺
6⁺
7⁺
12⁺
1⁻
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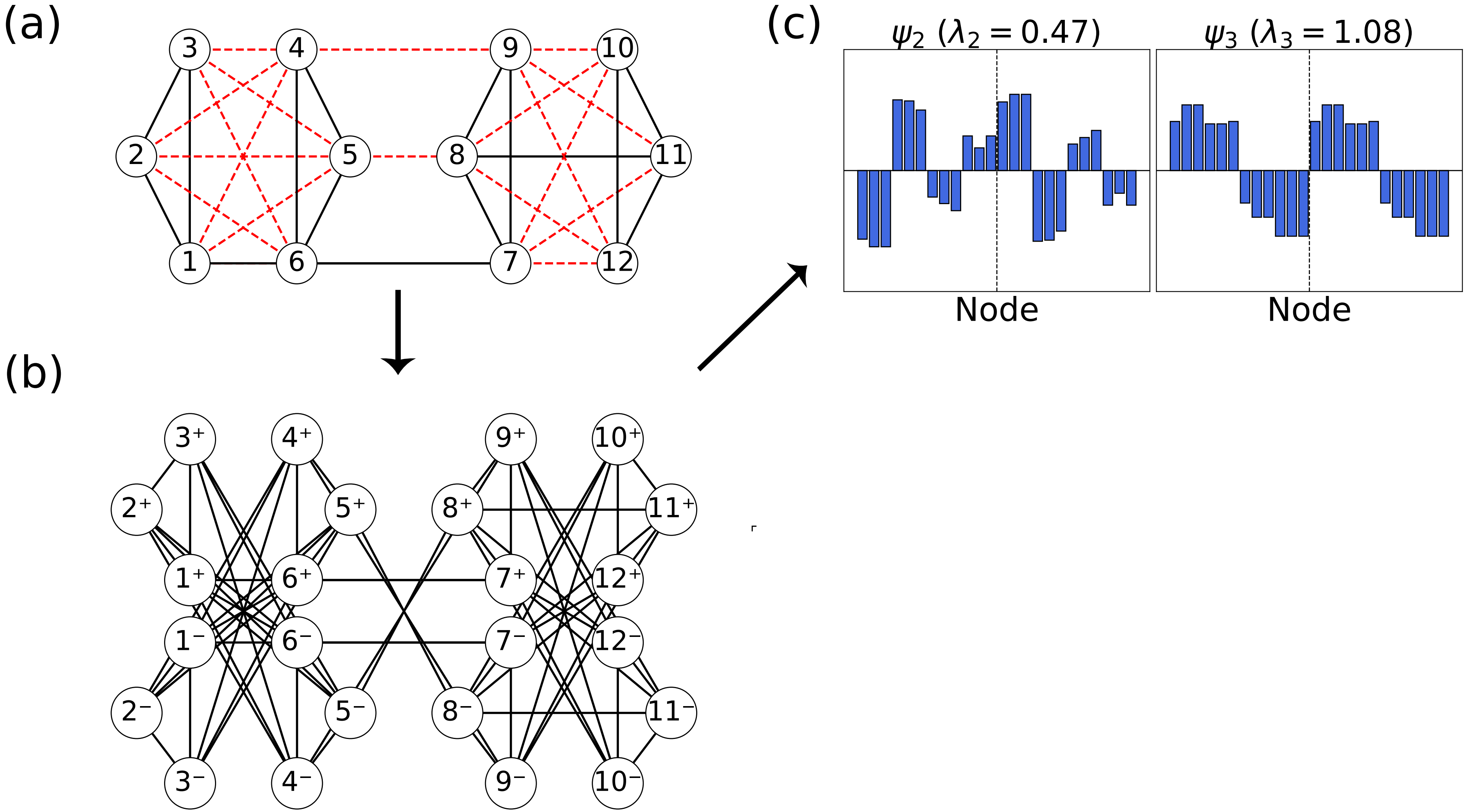
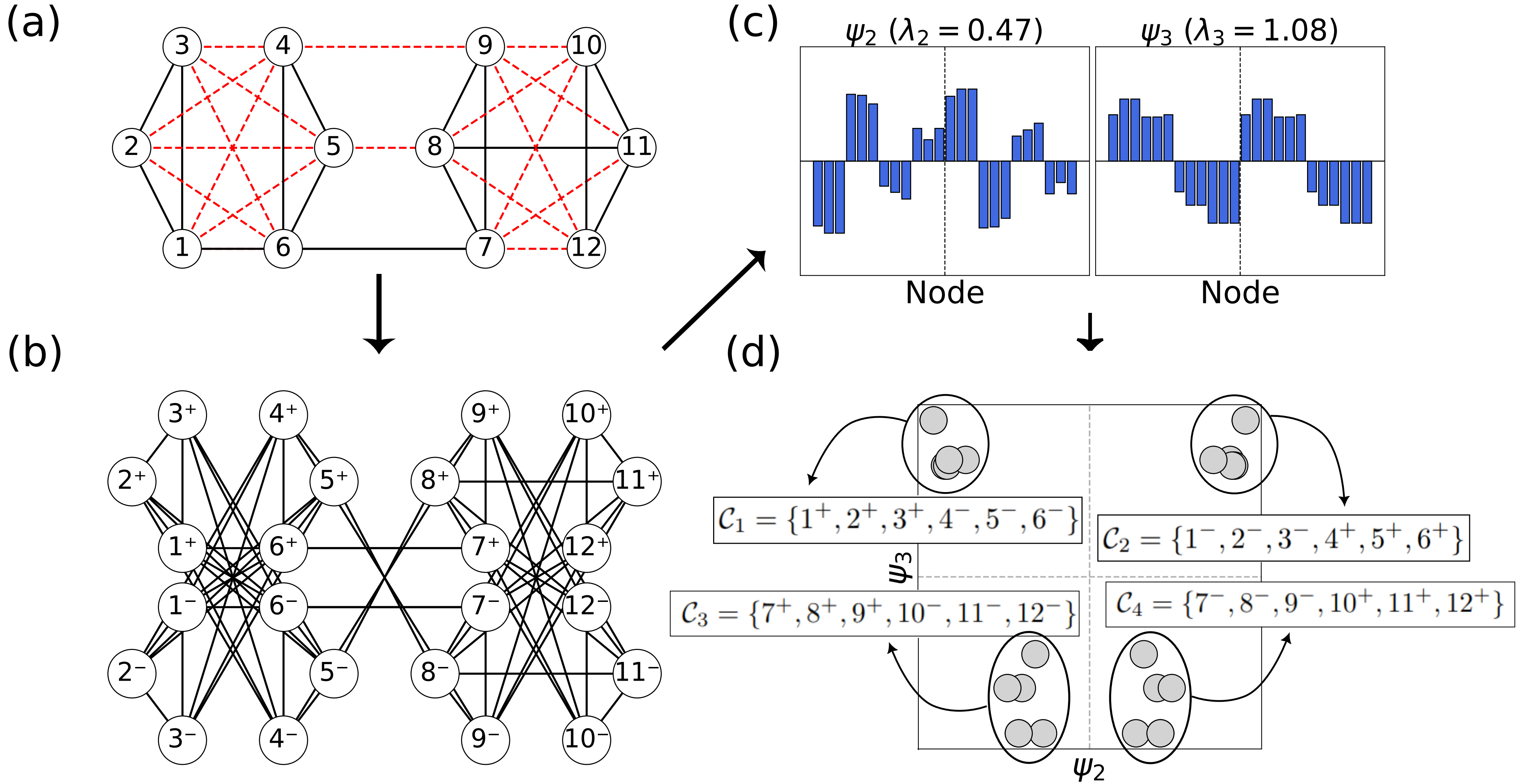
9⁻
10⁻



(a)
1
2
3
4
5
6
7
8
9
10
11
12
(b)
1⁺ 2⁺ 3⁺ 4⁺ 5⁺ 6⁺ 7⁺ 8⁺ 9⁺ 10⁺ 11⁺ 12⁺
1⁻ 2⁻ 3⁻ 4⁻ 5⁻ 6⁻ 7⁻ 8⁻ 9⁻ 10⁻ 11⁻ 12⁻
(c)
ψ_2 ($\lambda_2 = 0.47$)
ψ_3 ($\lambda_3 = 1.08$)
Node
Node

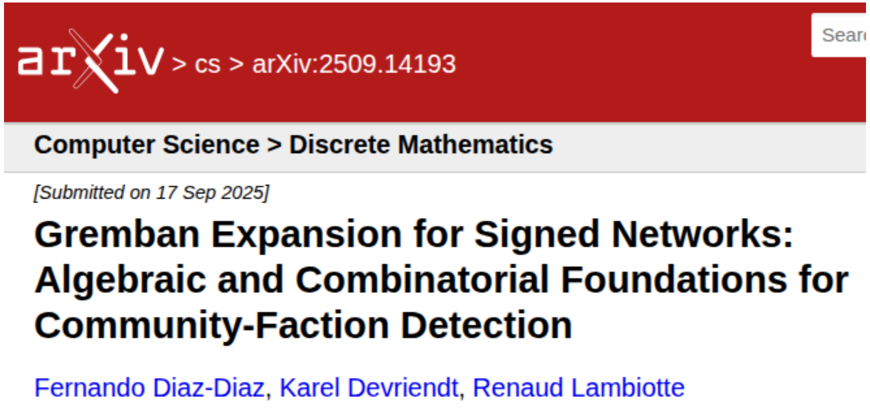


(a)
(b)
(c)
(d)
ψ_2 ($\lambda_2 = 0.47$)
ψ_3 ($\lambda_3 = 1.08$)
Node
Node
$\mathcal{C}_1 = \{1^+, 2^+, 3^+, 4^-, 5^-, 6^-\}$
$\mathcal{C}_2 = \{1^-, 2^-, 3^-, 4^+, 5^+, 6^+\}$
$\mathcal{C}_3 = \{7^+, 8^+, 9^+, 10^-, 11^-, 12^-\}$
$\mathcal{C}_4 = \{7^-, 8^-, 9^-, 10^+, 11^+, 12^+\}$
ψ_3
ψ_2

Gremban expansion: conclusions

- Signed graphs can show two mesoscale structures: **communities and factions.**
- Using the **Gremban expansion**, we map a signed graph to an unsigned one without losing sign information.
- Communities in the expanded graph map back to **communities or factions** in the original, depending on the **symmetry** of the partition.

Results avaliable in:

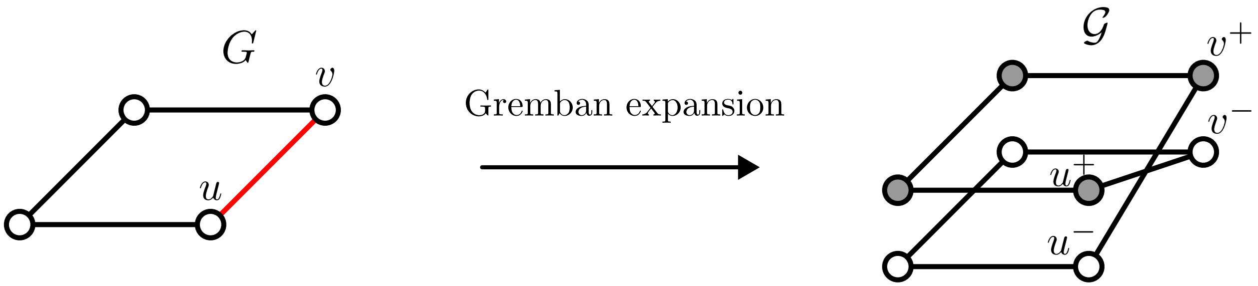

arXiv > cs > arXiv:2509.14193

Computer Science > Discrete Mathematics

[Submitted on 17 Sep 2025]

Gremban Expansion for Signed Networks: Algebraic and Combinatorial Foundations for Community-Faction Detection

Fernando Diaz-Diaz, Karel Devriendt, Renaud Lambiotte



Thank you!

Thank you for your attention

Contact: fddiaz@math.uc3m.es

Gremban expansion:

Signet review:

Signed Networks: theory, methods, and applications

Fernando Diaz-Diaz[1,2], Elena Candellone[3,4], Miguel A. González-Casado[1], Emma Fraxanet[5], Antoine Vendeville[6,7,8], Irene Ferri[9,10,11], and Andreia Sofia Teixeira[12,13]

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